Response of an Asymmetric Missile to Spin Varying through Resonance

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Asymptotic expansions of maximum response angle as well as resonance response angles are derived for constant spin acceleration and for spin acceleration to steady-state spin. These show that the important parameter to be considered is the ratio of missile damping rate to the square root of spin acceleration at resonance. Predictions of the approximate theory are compared with exact numerical calculation for a variety of missiles, and excellent agreement is seen for moderately damped missiles.

Nomenclature

A = $(\rho Sl^3/2I_y)(C_{m_0} + iC_{n_0})$
C_D = drag coefficient
$C_{L\alpha}$ = lift coefficient
$C_{M\alpha}$ = static moment coefficient
$C_{M_{p\alpha}}$ = Magnus moment coefficient
C_{m_0}, C_{n_0} = aerodynamic moment coefficients due to asymmetry
C_{l_δ} = roll-producing moment coefficient
C_{l_p} = roll damping moment coefficient
$C_{M_{\dot\alpha}}, C_{M_q}$ = damping moment coefficients
H = $(\rho Sl/2m)[C_{L\alpha} - C_D - k_t^{-2}(C_{M_q} + C_{M_{\dot\alpha}})]$
I_x, I_y = axial, transverse moments of inertia
k_a = axial radius of gyration $(I_x/ml^2)^{1/2}$
k_t = transverse radius of gyration $(I_y/ml^2)^{1/2}$
K_j = amplitude of the j mode ($j = 1,2$)
K_3 = amplitude of the response to asymmetric moment
K_δ = $(\rho Sl^3/2I_x)\delta_f C_{l_\delta}$
K_p = $-(\rho Sl/2m)[k_a^{-2}C_{l_p} + C_D]$
l = reference length
m = mass
M_X = axial component of aerodynamic moment
$M_{\tilde y}, M_{\tilde z}$ = transverse components of aerodynamic moment
M = $(\rho Sl/2m)k_t^{-2}C_{M\alpha}$
p = $d\phi/dt$, roll rate
P = $(I_x/I_y)(pl/V)$, gyroscopic spin
q = $(s - s_R)|\phi_R''/2|^{1/2}$
$\tilde q, \tilde r$ = angular velocity components along $\tilde Y$, $\tilde Z$ axis
s = dimensionless distance along flight path $\int_{t_0}^{t} (V/l)dt$
S = reference area
T = $(\rho Sl/2m)[C_{L\alpha} + k_a^{-2}C_{M_{p\alpha}}]$
V = magnitude of velocity
$X, \tilde Y, \tilde Z$ = nonrolling Cartesian coordinate axes
$\tilde\alpha$ = angle of attack
$\tilde\beta$ = angle of sideslip
δ_f = control surface deflection
θ = $2\bar K_p^{-2}[\exp(-\bar K_p q) + \bar K_p q - 1] = \phi - \bar\phi_1' q - \phi_R$
λ_j = damping rate of the j-modal amplitude K_j'/K_j
$\tilde\mu$ = $(\tilde q + i\tilde r)l/V$
ξ = $\tilde\beta + i\tilde\alpha$, complex angle of attack
ξ_A = trim response to forcing function
ρ = air density
ϕ = roll angle
ϕ_j = j-modal phase angle, $\phi_{j0} + \phi_j's$ (j = 1, 2)
ϕ_3 = response phase angle, $\phi_{30} + \phi$
ϕ_s' = K_δ/K_p, steady-state roll rate

Superscripts

$\sim$ = components in a nonrolling coordinate system
$(\bar{\ })$ = $(\ )|\phi_R''/2|^{-1/2}$
$(\ )'$ = primes denote derivatives with respect to s

Subscripts

0 = initial value
R = value at resonance

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Introduction

THE first concern of the designer of a finned missile is static stability. Once this is assured, his attention is directed toward avoiding the occurrence of spin rates near the pitch frequency of his particular missile. This spin-yaw resonance was first discussed by Nicolaides[1] in 1953 for linear moments and recently extended by various authors[2,3] to nonlinear moments. Although these studies were restricted by the assumption of constant spin rate, they did show that quite large trim angles can occur at resonance spin. Later papers[4,5] show that special roll moments can arise from various causes which cause the spin to "lock in" at the resonance value, and thus these large trim angles can occur frequently.

The effect of varying roll on the average trajectory was studied[6] in 1959 for very simple roll moments[7]: a constant roll-producing moment and a linear roll-damping moment. Although a number of computer calculations[8–10] have been made for the angular motion of a missile whose roll varies through resonance, very little theoretical work has been done on this more complicated motion. The related problem of a re-entering missile with varying pitch frequency is treated by Tolosko.[11] In his paper, resonance occurs when the varying pitch frequency equals the constant spin frequency. It is the purpose of this paper to obtain simple relations for the angular motion of a missile whose roll varies through resonance.

Equations of Motion

The rolling motion will be assumed to be caused by two simple roll moments: a roll-producing moment caused by control surface deflections δ_f and a roll-damping moment:

$$M_X = \tfrac{1}{2}\rho V^2 Sl[\delta_f C_{l_\delta} + (pl/V)C_{l_p}] \tag{1}$$

A very convenient independent variable is arc length along the trajectory, s. For this variable, the differential equation for the rolling motion becomes

$$\phi'' = K_\delta - K_p\phi' = K_p(\phi_s' - \phi') \tag{2}$$

where $\phi_s' = K_\delta/K_p$ is the steady-state roll rate, and the other

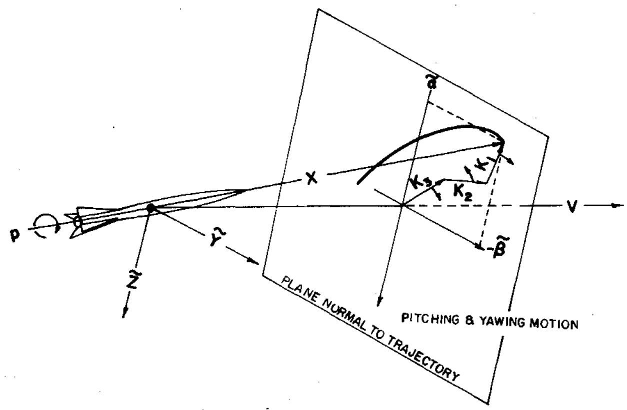

Fig. 1 Finned missile performing tricyclic motion.

symbols are defined in the Nomenclature. The solution to this differential equation for constant coefficients is[12]

$$\phi' = \phi_s' + (\phi_0' - \phi_s') \exp(-K_p s) \tag{3}$$

According to this equation, the roll rate has the initial value ϕ_0' but asymptotically approaches ϕ_s' as the missile travels along its flight path.

The pitch and yaw moments, given here in nonrolling coordinates $X,\tilde{Y},\tilde{Z}$, include the usual linear static, damping, and Magnus moments. Since resonance is important only when a missile-fixed constant-amplitude trim-producing moment is present, this essential moment term is included in our expansion. This trim-producing moment can be caused by center-of-gravity offset or by an inclined normal axis of inertia and unintentional configurational asymmetries, as well as by an intentional deflected control surface.[1,5] The moment we shall consider, then, takes on the form

$$M_{\tilde{y}} + iM_{\tilde{z}} = \tfrac{1}{2}\rho V^2 Sl\{[(pl/V)C_{M_{p\alpha}} - iC_{M_\alpha}]\xi + C_{M_q}\tilde{\mu} - iC_{M_{\dot{\alpha}}}\xi' + (C_{m_0} + iC_{n_0})e^{i\phi}\} \tag{4}$$

where $\xi = \tilde{\beta} + i\tilde{\alpha}$ is the complex angle of attack, and $\tilde{\mu} = (\tilde{q} + i\tilde{r})l/V$ is the complex transverse angular velocity.

The only forces we shall consider are drag and a linear lift force. For these forces and moments and for small geometrical angles, the differential equation of motion for the complex angle of attack can be written in a fairly simple form:

$$\xi'' + (H - iP)\xi' - (M + iPT)\xi = iAe^{i\phi} \tag{5}$$

Constant Roll Rate

Equation (5) can be solved easily for constant spin rate; the solution is the well-known equation for tricyclic motion (Fig. 1)[1,12]:

$$\xi = K_1 \exp(i\phi_1) + K_2 \exp(i\phi_2) + K_3 \exp[i(\phi_{30} + \phi)] \tag{6}$$

where

$$K_j = K_{j0} \exp(\lambda_j s) \qquad j = 1,2$$
$$\lambda_j = -(H\phi_j' - PT)(2\phi_j' - P)^{-1}$$
$$\phi_j' = \tfrac{1}{2}[P \pm (P^2 - 4M)^{1/2}]$$

The magnitude and initial phase angle for the trim mode, K_3 and ϕ_{30}, can be found by direct substitution in Eq. (5):

$$K_3 \exp(i\phi_{30}) = -iA/[(\phi')^2 - P\phi' + M - i(\phi'H - PT)] \tag{7}$$

For zero spin, the magnitude of the trim takes on a very simple form:

$$K_{30} = |A/M| = |(C_{m_0} + iC_{n_0})/C_{M_\alpha}| \tag{8}$$

As can be seen from Eq. (7), the trim amplitude increases from the zero-spin value to a maximum at resonance and then decays to zero as spin increases to large values. Resonance occurs when $\phi' = \phi_j'$, and the real part of the denominator of Eq. (7) becomes zero. For a statically stable missile, the natural frequencies are opposite in sign, but usually the frequency with the sign of the spin is ϕ_1'. This frequency is the larger in magnitude of the two and is sometimes called the nutation rate. Throughout this paper, we shall assume the spin to be positive for simplicity. The modifications to some of the results for negative spin are quite simple.

The resonance amplitude of the trim can now be given:

$$K_{3R} = |A[\lambda_1\phi_1'(2 - I_x/I_y)]^{-1}| \tag{9}$$

The ratio of the moments of inertia is usually less than 0.1 and therefore can be neglected in comparison with 2. This approximation also implies that $|\phi_1'| = |\phi_2'|$ when spin is near resonance. With this in mind, we can obtain a very simple relation for the resonance amplification of the zero-spin trim angle:

$$K_{3R}/K_{30} = |\phi_1'/2\lambda_1| \tag{10}$$

A sample plot of the variation of trim amplitude for constant spin rate with the value of the spin rate is given in Fig. 2. As can be seen from this figure, quite large magnification factors are possible. If the spin varies rapidly through resonance, some fraction of these values should be observed. Our objective is to find the relations between the angles actually produced and the corresponding roll accelerations.

Varying Roll Rate

If the spin is nonconstant, Eq. (5) is a linear equation with variable coefficients and a specified forcing function. The solutions[12] for the two transient motions now have varying frequencies, which are given by the relation after Eq. (6), but their damping rates are changed:

$$\lambda_j = -(H\phi_j' - PT + \phi_j'')(2\phi_j' - P)^{-1} \tag{11}$$

The usual size for H is 10^{-3}, whereas M and T are 10^{-4}. Near resonance, P is 10^{-3} or less. As can be seen from the relations for ϕ_j' and λ_j, P has a small effect. We shall approximate P by its value for resonance, but we could make the simpler approximation that P is zero. For either approximation, a particular integral for arbitrary forcing functions can be obtained by the method of variation of parameters[12]:

$$\xi_A = A(\phi_1' - \phi_2')^{-1}\left\{\exp(\lambda_1 s + i\phi_1' s)\int_0^s f_1(\hat{s})\, d\hat{s} - \exp(\lambda_2 s + i\phi_2' s)\int_0^s f_2(\hat{s})\, d\hat{s}\right\} \tag{12}$$

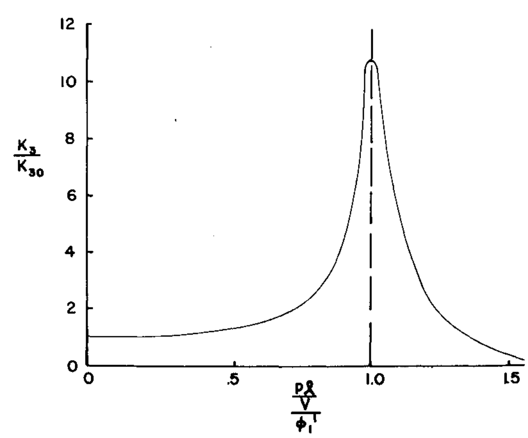

Fig. 2 Constant spin response to aerodynamic asymmetry.

where

$$\tilde{\xi}_A'(0) = \tilde{\xi}_A(0) = 0$$
$$f_j(\hat{s}) = \exp\{-\lambda_j\hat{s} + i[\phi(\hat{s}) - \phi_j'\hat{s}]\}$$
$$\lambda_j \leqq 0$$

Near resonance, the two integrals in Eq. (12) have quite different character. The integrand for the second integral has a frequency of $2\phi_1'$, whereas that for the first has a zero frequency and, therefore, a "stationary phase." The second integral is much smaller than the first and can be replaced by the first term in an asymptotic expansion[13] in the resonance value of the roll acceleration ϕ_R'':

$$\int_0^s f_2(\hat{s})\, d\hat{s} = i[(\phi'(0) - \phi_2')^{-1}e^{i\phi(0)} - \{\phi'(s) - \phi_2'\}^{-1}f_2(s)] + o(|\phi_R''|^{-3/2}) \quad (13)$$

The origin for s is taken to be a long distance before the location at which resonance occurs, so that the transient part of $\tilde{\xi}_A$ can damp out and $\tilde{\xi}_A$ gives us the trim response to the forcing function.

The first integral can be simplified by shifting the origin to the point at which resonance occurs, s_R, and rescaling using ϕ_R'':

$$q = (s - s_R)|\phi_R''/2|^{1/2} \quad (14)$$

Therefore

$$\tilde{\xi}_A = \bar{A}(\phi_1' - \phi_2')^{-1}\Big\{\exp[(\bar{\lambda}_1 + i\bar{\phi}_1')q] \times \int_{q_0}^q \exp(-\bar{\lambda}_1\hat{q} + i\theta)d\hat{q} + i(\bar{\phi}' - \phi_2')^{-1} \times \exp[i(\phi - \phi_R)]\Big\}\exp(i\phi_R) \quad (15)$$

$$\theta \equiv \phi - \bar{\phi}_1'q - \phi_R = 2\bar{K}_p^{-2}[\exp(-\bar{K}_pq) + \bar{K}_pq - 1] \quad (16)$$

where

$$(\bar{\ }) = (\ )|\phi_R''/2|^{-1/2}$$

For the simple case of constant roll acceleration, $K_p = 0$, $\phi_R'' = K_\delta$, and $\theta = q^2$. The scale factor in Eq. (14) was selected to give this simple result.

We can now replace the large negative number q_0 by $-\infty$ in the lower limit of the integral, omit the small term added to the integral, and divide by K_{3R} to obtain a good approximate relation for the ratio of the actual variable spin trim angle to the resonance trim angle:

$$|\tilde{\xi}_A|K_{3R}^{-1} = \left|\bar{\lambda}_1\exp(\bar{\lambda}_1q)\int_{-\infty}^q \exp(-\bar{\lambda}_1\hat{q} + i\theta)\, d\hat{q}\right| \quad (17)$$

Equation (17) shows that the most important parameter of the problem is $\bar{\lambda}_1 = \lambda_1|\phi_R''/2|^{-1/2}$. This is the only parameter for constant roll acceleration. If varying roll acceleration is considered, the parameter $\bar{K}_p = \bar{K}_p|\phi_R''/2|^{-1/2}$ appears as a secondary parameter.

Asymptotic Expansion of Eq. (17)

Asymptotic expansions of integrals of the form appearing in Eq. (17) can be obtained by the method of stationary phase.[13] According to this theory, the integral can be expressed in terms of functions of the integrand evaluated at the endpoints and those points at which θ'' vanishes. The results can best be stated by splitting the integral into an integral from $-\infty$ to 0 and one from 0 to q. If terms up to but not including those of order $|\phi_R''|^{-3/2}$ are retained, Ref.

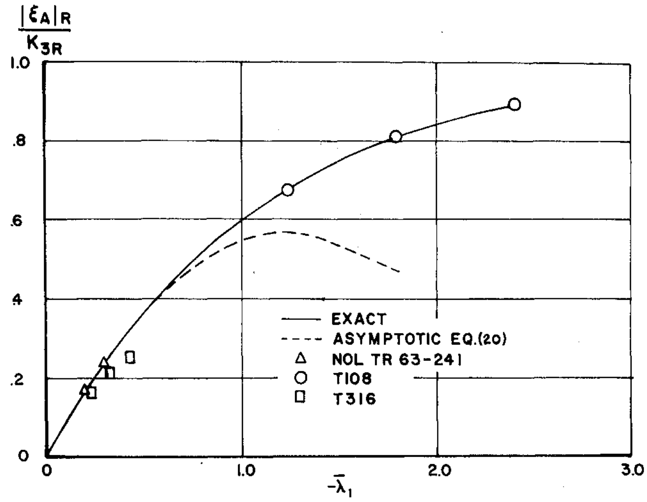

Fig. 3 Varying spin response at resonance vs ratio of damping to roll acceleration, $\bar{\lambda}_1 = \lambda_1|\phi_R''/2|^{-1/2}$.

13 provides us with the following relations for $\phi_R'' > 0$:

$$\int_{-\infty}^0 \exp(-\bar{\lambda}_1\hat{q} + i\theta)\, d\hat{q} = \left(\frac{\pi^{1/2}}{2}\right)[(1 + ia)e^{\pi i/4} + ib] \quad (18)$$

$$\int_0^q \exp(-\bar{\lambda}_1\hat{q} + i\theta)\, d\hat{q} = \left(\frac{\pi^{1/2}}{2}\right)[(1 + ia)e^{\pi i/4} - ib] - \frac{i\exp[-\bar{\lambda}_1q + i\theta(q)]}{\bar{\theta}'(q)} \quad (19)$$

where

$$a = [\bar{\lambda}_1^2 + \bar{\lambda}_1\bar{K}_p + \tfrac{1}{6}\bar{K}_p^2]/4$$
$$b = [\bar{\lambda}_1 + \tfrac{1}{3}\bar{K}_p]\pi^{-1/2}$$

If ϕ_R'' is negative, the right sides of Eqs. (18) and (19) should be replaced by their complex conjugates.

Constant Roll Acceleration

We shall now consider the special case of constant roll acceleration in some detail. Equations (17) and (18) can be used to give an approximate relation for $|\tilde{\xi}_A|$ at resonance:

$$|\tilde{\xi}_A|_RK_{3R}^{-1} = (\pi^{1/2}/2)|[\bar{\lambda}_1 + (i\bar{\lambda}_1^3/4)]e^{\pi i/4} + i\bar{\lambda}_1^2\pi^{-1/2}| \quad (20)$$

For $q = 0$, the integral in Eq. (17) can be computed exactly in terms of Fresnel integrals.[14] The results are compared with Eq. (20) in Fig. 3. We see reasonably good agreement for $-\bar{\lambda}_1$ less than 0.6.

After the spin rate accelerates through resonance, the response angle continues to grow. The asymptotic relation now becomes

$$\tilde{\xi}_A|K_{3R}^{-1} = |\pi^{1/2}[\bar{\lambda}_1 + i\bar{\lambda}_1^3/4]\exp(\bar{\lambda}_1q) + \bar{\lambda}_1\exp[i(q^2 - 3\pi/4)]/2q| \quad (21)$$

This relation is not good near resonance, since the second term becomes infinite. However, we see that near $q^2 = 3\pi/4$ the two terms are additive, and a maximum should exist near there.

The exact location of this maximum can be found by differentiating Eq. (17), setting the result equal to zero, and finding the root close to $q^2 = 3\pi/4$. A curve of $q_{\max}$ vs $-\bar{\lambda}_1$ is obtained by this method and plotted in Fig. 4. In Fig. 5, the corresponding curve of $|\tilde{\xi}_A|_{\max}$ is plotted vs $-\bar{\lambda}_1$ by use of Eq. (17). The values of $q_{\max}$ are also used in the approximate Eq. (21) and the result given in Fig. 5. The good agreement of these two curves is quite encouraging.

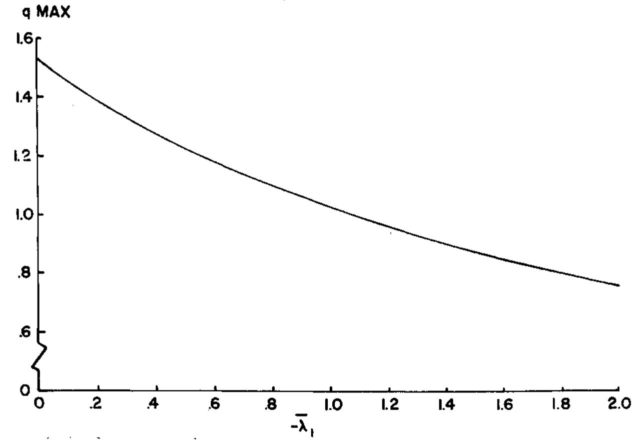

Fig. 4 Location of maximum varying spin response vs $\bar{\lambda}_1$.

Equation (21) gives only the magnitude of the angular motion after resonance. If we use the asymptotic relations for Eq. (15), the variation in orientation of the total angle of attack can be seen:

$$\xi_A = \bar{A}\{(\phi_1' - \phi_2')^{-1}\pi^{1/2}[1 + i(\bar{\lambda}_1^2/4)] \times \exp[(\bar{\lambda}_1 + i\bar{\phi}_1')q + i(\pi/4 + \phi_R)] - |\phi_R''/2|^{1/2} i e^{i\phi}(\phi' - \phi_1')^{-1}(\phi' - \phi_2')^{-1}\} \quad (22)$$

The second term in Eq. (22) is the usual trim angle for constant spin rate. The effect of the spin varying through resonance is to add a disturbance that rotates at the fast frequency, *not the spin frequency,* and that is damped at the damping rate of the fast frequency. The amplitude of this disturbance is a function of the spin acceleration and can be computed approximately by Eq. (22) or exactly by Eq. (17).

Comparison with Exact Calculations

In Refs. 9 and 10, a number of computations of Eq. (5) are done for actual missiles. The particular parameters for these calculations are given in Table 1 and plotted in Figs. 3 and 5. Here again, the agreement is good. As can be seen from the table, values of $\bar{K}_p$ are not zero for Ref. 9 results, but they are small compared with corresponding values of $\bar{\lambda}_1$. Thus we can compare these cases with the predictions for constant spin acceleration. The discrepancies for these results, however, may be due to the nonzero values of $\bar{K}_p$.

The T108 and T316 are quite different missiles, and their behavior as given in Ref. 9 appears to be unrelated. Figure 5 shows that these results can be related by the parameter $\bar{\lambda}_1$. The T108 has high values of this parameter and achieves over 90% of K_{3R}, whereas the T316 has much lower values and achieves about 40–60% of K_{3R}.

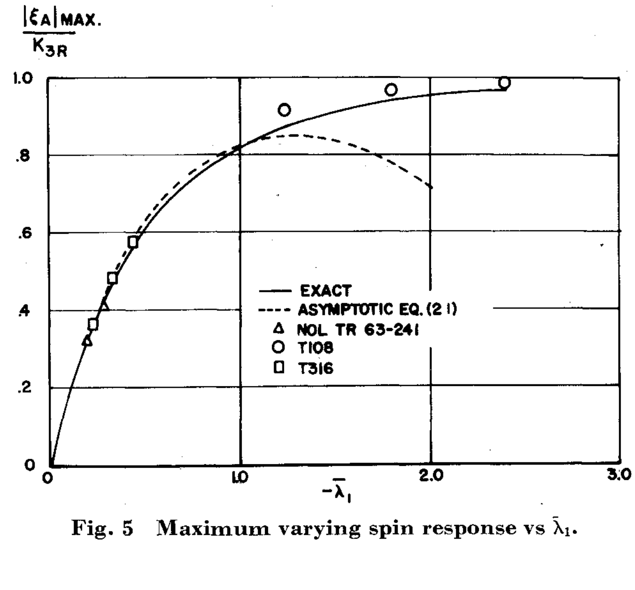

Fig. 5 Maximum varying spin response vs $\bar{\lambda}_1$.

Table 1 Parameters used in exact calculations

Case	$\bar{K}_p$	$\bar{\lambda}_1$	$\bar{\phi}_1'$	$\lvert\xi_A\rvert_R K_{3R}^{-1}$	$\lvert\xi_A\rvert_{\max} K_{3R}^{-1}$
T108	0.05	−1.25	9.6	0.68	0.91
T108	0.08	−1.82	14.0	0.81	0.96
T108	0.11	−2.44	18.8	0.89	0.99
T316	0.04	−0.23	15.3	0.16	0.36
T316	0.06	−0.34	22.4	0.21	0.48
T316	0.08	−0.44	29.0	0.25	0.58
NOL	0	−0.20	7.3	0.17	0.32
NOL	0	−0.29	10.3	0.24	0.41

Spin Lock-In

It is a very common occurrence for a missile spinning through resonance to "lock in" at resonance spin rate for some distance along its flight path. For this case,

$$\theta_{\text{lock-in}} = \theta(q) \qquad q < 0$$
$$= 0 \qquad 0 < q < q^* \qquad (23)$$
$$= \theta(q - q^*) \qquad q^* < q$$

where q^* is the point on the flight path at which the spin rate breaks out of "lock-in," and $\theta(q)$ is given by Eq. (16). The magnitude of the angular motion can then be computed from Eq. (17) for this rolling motion:

$$|\xi_A|K_{3R}^{-1} = \left|\bar{\lambda}_1 \exp(\bar{\lambda}_1 q) \int_{-\infty}^{0} \exp(-\bar{\lambda}_1 q + i\theta)\, dq + \exp(\bar{\lambda}_1 q) - 1\right| \qquad 0 < q < q^* \quad (24)$$

$$|\xi_A|K_{3R}^{-1} = \exp(\bar{\lambda}_1 q)\left|\bar{\lambda}_1 \int_{-\infty}^{0} \exp(-\bar{\lambda}_1 \hat{q} + i\theta)\, d\hat{q} + \bar{\lambda}_1 \exp(-\bar{\lambda}_1 q^*) \int_{0}^{q-q^*} \exp(-\bar{\lambda}_1 \hat{q} + i\theta)\, d\hat{q} + 1 - \exp(-\bar{\lambda}_1 q^*)\right| \qquad q^* < q \quad (25)$$

The integrals in Eqs. (24) and (25) can be easily approximated by asymptotic expressions from Eqs. (18) and (19). The value of q^* unfortunately depends on the nonlinear roll lock-in moments and is quite difficult to estimate.

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Response of Flight Vehicles to Nonstationary Atmospheric Turbulence

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Exploratory studies are made on the feasibility of using the theory of nonstationary random processes in the vehicle-response analysis. To this end, response statistics are computed for the plunging rigid body motion of the vehicle traveling in a nonstationary turbulence field. The approach is an adaptation of Priestley's evolutionary spectral analysis. Further assuming that the turbulence field is Gaussian distributed expressions for the ensemble average of the threshold crossing rate and the probability density of response peak magnitude conditional on the occurrence of a peak are derived. Numerical examples presented in the paper clearly demonstrate that a nonstationary analysis is not only feasible but it also rectifies an unconservative aspect of the traditional stationary approach which cannot account for possible transient overloads.

Introduction

THE statistical approach to the dynamic analysis of flight vehicles subjected to atmospheric turbulence usually presupposes that the turbulence field is homogeneous and isotropic.[1] Then, in the simplest idealization, where a flight vehicle is treated as a single-degree-of-freedom system, (considering only the plunging rigid body motion) the forcing function in the governing differential equation is a stationary random process. The mean square value of the steady-state system response can be obtained, therefore, from an integration over frequency of the product of the spectral density of the gust velocity and the squared absolute value of an appropriate frequency-response function. Many such analyses are available in the literature.[2] Two commonly used spectral densities for the gust velocity are the Dryden spectrum and the von Kármán spectrum. Also, the gust velocity is generally assumed to be a Gaussian random process for the convenience of computing the statistics of threshold crossing and peak magnitude of the response.

It has been verified experimentally[3,4] that the gust velocity is approximately Gaussian distributed.‡ However, strong nonstationarity characteristics are evident in some cases, especially, for low-altitude turbulence over rough terrain. That a nonstationary analysis is needed is further evidenced by the recent treatment of the mean square gust velocity as a random variable.[6]

The purpose of this paper is to explore the feasibility of modeling atmospheric turbulence by a nonstationary random process in the vehicle response analysis. Sample calculations will be presented on the statistics of the response including threshold crossings and peak distribution. In order that the main features of the nonstationary analysis not be obscured by computational details, we shall restrict our discussions to a single-degree-of-freedom system and to two-dimensional incompressible flow as in the pioneering stationary analysis of Fung.[2] It is interesting to note, however, that such simplifications are commonly made at the preliminary design stage even for large flight vehicles. For the reader's convenient reference, some aspects of the random process theory required for the present analysis are reviewed in the Appendices.

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Index categories: Aircraft Gust Loading and Wind Shear; Aircraft Vibration; Structural Dynamic Analysis.

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‡ Theoretically, the turbulence velocity can not be exactly Gaussian distributed.[5]